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A Study on State Constraint Optimal Control Problems

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Abstract

In this paper, we use partial differential equations to deal with constraint optimal control problems. We construct extremal flows by differential-algebraic equation to approximate the optimal objective value of constraint optimal control problems. We prove a convergent theorem for an approximation approach to the optimal objective value of a state-constraint optimal control problem.

Subject Areas

Mathematics

Keywords

State Constraint Optimal Control, Partial Differential Equations, Extremal Flow, Differential-Algebraic Equation

1. Introduction

To solve an optimal control problem, it is a rather common procedure to use a direct discretization approach to exact solution for the problem ([1]-[3]). One usually expects a desired error between a numerical solution and the infimum of the original problem. However, a direct discretization method is capable of solving some constrained optimal control problem efficiently from an engineering point of view, but there is no theoretical foundation in terms of a convergence result for these methods as yet. For this issue, the authors ([3]) have given an analysis in detail. We deal with state constrained optimal control problems considered in paper by a new mathematical method. The main purpose of this paper is to introduce an extremal flow by a partial differential equation for an constraint optimal control problem to ensure the convergence process before implementing a numerical process.

We consider two constraint optimal control problems as follows:

1)

$$(\mathcal{P}_1):\begin{cases}\text{Minimize } P(x(T)),\\ \text{s.t. } \dot{x}(t)=Ax(t)+Bu(t),\ x(0)=a\in R^n,\\ \quad c^{\mathrm{T}}x(t)+\beta u^{\mathrm{T}}(t)u(t)\le 0,\\ \quad u(t)\in R^m,\ x(t)\in R^n,\ t\in[0,T]\end{cases} \tag{1.1}$$

where $\beta>0$ is given and the cost function $P(x)$ is continuously differentiable on R^n. The matrices $A\in R^{n\times n}$, $B\in R^{n\times m}$ and the vector $c\in R^n$ are given in the linear control system in (1.1). Where $\beta>0$ is given and the cost function $P(x)$ is continuously differentiable on R^n. The matrices $A\in R^{n\times n}$, $B\in R^{n\times m}$ and the vector $C\in R^n$ are given in the linear control system in (1.1).

2)

$$(\mathcal{P}_2):\begin{cases}\text{Minimize } P(x(T)),\\ \text{s.t. } \dot{x}(t)=Ax(t))+Bu(t),\ x(0)=a\in R^n,\\ \quad c^{\mathrm{T}}x(t)\le 0,\\ \quad u(t)\in R^n,\ x(t)\in R^n,\ t\in[0,T]\end{cases} \tag{1.2}$$

where $M>0$ is given and the cost function $P(x)$ is continuously differentiable on R^n. The matrices $A\in R^{n\times n}$, $B\in R^{n\times n}$ and the vector $c\in R^n$ are given in the linear control system in (1.2).

We suppose that the matrices A,B and the vectors c,a appearing in these problems above are the same. But when we consider the problem $(\mathcal{P}_2)$ by means of $(\mathcal{P}_1)$, we assume $m=n$ for $(\mathcal{P}_1)$.

The rest of the paper is organized as follows. In Section 2, to deal with the problem $(\mathcal{P}_1)$, we present a partial differential equation by rewriting HJB equation. Then we create an extremal flow by a differential-algebraic equation to give an optimal feedback control for the problem $(\mathcal{P}_1)$. In Section 3, we prove a convergent theorem for an approximation approach to the optimal objective value of the problem $(\mathcal{P}_2)$ by optimal objective values of a series of problem $(\mathcal{P}_1)$.

2. On Optimal Control Problem $(\mathcal{P}_1)$ with a Partial Differential Equation

In this section, we deal with the optimal control problem $(\mathcal{P}_1)$ by a partial differential equation. In the following, the positive number β is fixed. For given $x\in R^n$, define a set

$$S_1(x)=\left\{u:u^{\mathrm{T}}u\le\frac{-c^{\mathrm{T}}x}{\beta}\right\}. \tag{2.1}$$

Note that if $c^{\mathrm{T}}x>0$ then $S_1(x)=\varnothing$. In the following, we assume that

$$S_1(x)\ne\varnothing. \tag{2.2}$$

We consider the Hamilton-Jacobi-Bellman equation for optimal control theory

([2]) as follows:

$$v_t(t,x)+v_x^{\mathrm{T}}(t,x)Ax+\min_{u\in S_1(x)}\{v_x^{\mathrm{T}}(t,x)Bu\}=0,\ v(T,x)=P(x). \tag{2.3}$$

For given $(\lambda,x)\in R^m\times R^n$, define a function

$$H(\lambda,x):=\min_{u\in S_1(x)}\{\lambda^{\mathrm{T}}u\}, \tag{2.4}$$

then given $(t,x)\in R^1\times R^n$, for $\lambda=B^{\mathrm{T}}v_x(t,x)$, we have

$$H(B^{\mathrm{T}}v_x(t,x),x)=\min_{u\in S_1(x)}\{v_x^{\mathrm{T}}(t,x)Bu\}, \tag{2.5}$$

By (2.5), we can rewrite the Hamilton-Jacobi-Bellman equation in (2.3) as a partial differential equation as follows ([4] [5]):

$$v_t(t,x)+v_x^{\mathrm{T}}(t,x)Ax+H(B^{\mathrm{T}}v_x(t,x),x)=0,\ v(T,x)=P(x). \tag{2.6}$$

We will deal with the optimal control problem $(\mathcal{P}_1)$ in (1.1) by the partial differential equation in (2.6).

Given a pair $(x,u)\in R^n\times R^m$, by the definition of $S_1(x)$, the relationship $u\in S_1(x)$ means

$$u^{\mathrm{T}}u\le\frac{-c^{\mathrm{T}}x}{\beta}.$$

Then by primary optimization, for a nonzero vector $\lambda\in R^m$,

$$\min_{u\in S_1(x)}\{\lambda^{\mathrm{T}}u\}=\min_{u^{\mathrm{T}}u\le\beta^{-1}(-c^{\mathrm{T}}x)}\{\lambda^{\mathrm{T}}u\}=-\sqrt{\beta^{-1}(-c^{\mathrm{T}}x)}\|\lambda\|. \tag{2.7}$$

By (2.4), (2.5), (2.7), we have,

$$H(B^{\mathrm{T}}v_x(t,x),x)=\min_{u\in S_1(x)}\{v_x^{\mathrm{T}}(t,x)Bu\}=-\sqrt{\beta^{-1}(-c^{\mathrm{T}}x)}\|B^{\mathrm{T}}v_x(t,x)\|. \tag{2.8}$$

Remark 2.1. By (2.8), we can rewrite the partial differential equation in (2.6) as

$$v_t(t,x)+v_x^{\mathrm{T}}(t,x)Ax-\sqrt{\beta^{-1}(-c^{\mathrm{T}}x)}\|B^{\mathrm{T}}v_x(t,x)\|=0,\ v(T,x)=P(x). \tag{2.9}$$

Let $v(t,x)$ be a solution of the partial differential equation in (2.9). If for $c^{\mathrm{T}}x\le 0$ $v_x(t,x)$ is continuous, then $H(B^{\mathrm{T}}v_x(t,x),x)=-\sqrt{\beta^{-1}(-c^{\mathrm{T}}x)}\|B^{\mathrm{T}}v_x(t,x)\|$ is continuous on $\{x:c^{\mathrm{T}}x\le 0\}$. By the viscosity solution of optimal control theory, there is a viscosity approximation to the solution of the above partial differential equation ([6]-[9]).

Definition 2.1. For a solution $v(t,x)$ of the partial differential equation in (2.9), we call $(\hat{x}(\cdot),\hat{u}(\cdot))$ an extremal flow related to $v(t,x)$ if it is a solution of the following differential-algebraic equation:

$$\begin{cases}\dot{\hat{x}}(t)=A\hat{x}(t)+B\hat{u}(t),\\ v_x^{\mathrm{T}}(t,\hat{x}(t))B\hat{u}(t)=-\sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}\|B^{\mathrm{T}}v_x(t,\hat{x}(t))\|,\\ x\in[0,T],\ \hat{x}(0)=a,\end{cases} \tag{2.10}$$

Theorem 2.1. Let $v(t,x)$ be a solution of the partial differential equation in (2.9) and $(\hat{x}(\cdot),\hat{u}(\cdot))$ be an extremal flow defined by (2.10). Then, $\hat{u}(\cdot)$ is an optimal control of the problem $(\mathcal{P}_1)$, $P(\hat{x}(T))$ is the optimal value of the problem $(\mathcal{P}_1)$ and $P(\hat{x}(T)) = v(0,a)$.

Proof. By (2.10), we have

$$v_x^{\mathrm{T}}(t,\hat{x}(t))B\hat{u}(t) = -\sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}\left\|B^{\mathrm{T}}v_x(t,\hat{x}(t))\right\|. \tag{2.11}$$

By (2.9), (2.10), (2.11), we have, for $t \in [0,T]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v(t,\hat{x}(t)) &= v_t(t,\hat{x}(t)) + v_x^{\mathrm{T}}(t,\hat{x}(t))A\hat{x}(t) + v_x^{\mathrm{T}}(t,\hat{x}(t))B\hat{u}(t)\\ &= v_t(t,\hat{x}(t)) + v_x^{\mathrm{T}}(t,\hat{x}(t))A\hat{x}(t) - \sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}\left\|B^{\mathrm{T}}v_x(t,\hat{x}(t))\right\|\\ &= 0.\end{aligned} \tag{2.12}$$

Integrating the above equality with respect to t from 0 to T, noting $v(T,\hat{x}(T)) = P(\hat{x}(T)), \hat{x}(0) = a$, we have

$$0 = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}v(t,\hat{x}(t))\mathrm{d}t = P(\hat{x}(T)) - v(0,a). \tag{2.13}$$

Let $(x(\cdot),u(\cdot))$ be an arbitrary admissible pair of the control system in the problem $(\mathcal{P}_1)$. We have, for $t \in [0,T]$,

$$c^{\mathrm{T}}x(t) + \beta u^{\mathrm{T}}(t)u(t) \le 0, \tag{2.14}$$

which implies $u(t) \in S(x(t))$. Thus, by (2.5), (2.7) with $\lambda = B^{\mathrm{T}}v_x(t,x(t))$ for a $t \in [0,T]$, we have

$$-\sqrt{-\beta^{-1}c^{\mathrm{T}}x(t)}\left\|B^{\mathrm{T}}v_x(t,x(t))\right\| \le v_x^{\mathrm{T}}(t,x(t))Bu(t). \tag{2.15}$$

Then for each $t \in [0,T]$, by the partial deferential equation in (2.9), also noting that $(x(\cdot),u(\cdot))$ is an arbitrary admissible pair of the control system in the problem $(\mathcal{P}_1)$, we have

$$\begin{aligned}0 &= v_t(t,x(t)) + v_x^{\mathrm{T}}(t,x(t))Ax(t) - \sqrt{-\beta^{-1}c^{\mathrm{T}}x(t)}\left\|B^{\mathrm{T}}v_x(t,x(t))\right\|\\ &\le v_t(t,x(t)) + v_x^{\mathrm{T}}(t,x(t))Ax(t) + v_x^{\mathrm{T}}(t,x(t))Bu(t)\\ &= v_t(t,x(t)) + v_x^{\mathrm{T}}(t,x(t))\frac{\mathrm{d}x(t)}{\mathrm{d}t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}v(t,x(t)).\end{aligned} \tag{2.16}$$

Integrating the above inequality over $[0,T]$, noting $v(T,x(T)) = P(x(T))$, $x(0) = a$, by (2.16), we have

$$0 \le \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}v(t,x(t))\mathrm{d}t = P(x(T)) - v(0,a). \tag{2.17}$$

By (2.13), (2.17), we have

$$P(\hat{x}(T)) \le P(x(T)). \tag{2.18}$$

By (2.18), we see that $P(\hat{x}(T))$ is the optimal value of the problem $(\mathcal{P}_1)$ and $\hat{u}(\cdot)$ is an optimal control to the problem $(\mathcal{P}_1)$. But by (4.8), we have $P(\hat{x}(T)) = v(0,a)$. The theorem has been proved.

Let $(\hat{x}(\cdot),\hat{u}(\cdot))$ be an extremal flow. By the definition of extremal flow in

Definition 2.1, we have

$$v_x^{\mathrm{T}}\left(t,\hat{x}(t)\right)B\hat{u}(t)=-\sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}\left\|B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\right\|,\ \left(t\in[0,T]\right). \tag{2.19}$$

We have reached the following result.

Theorem 2.2. Let $v(t,x)$ be a solution of the partial differential equation in (2.9) and $\left(\hat{x}(\cdot),\hat{u}(\cdot)\right)$ be an extremal flow. If, for some $t\in[0,T]$, $B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\neq 0$, then

$$\hat{u}(t)=-\sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}\frac{B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)}{\left\|B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\right\|}. \tag{2.20}$$

Proof. Since $B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\neq 0$, we see by (2.19) that the vector $\hat{u}(t)$ is in an opposite direction to $B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)$. It follows from the inner product on the left side of (2.19) that $\left\|\hat{u}(t)\right\|=\sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}$. Then we have

$$\hat{u}(t)=-\sqrt{-\beta^{-1}c^{\mathrm{T}}\hat{x}(t)}\frac{B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)}{\left\|B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\right\|}.$$

The theorem has been proved.

Remark 2.2. Let $v(t,x)$ be a solution of the partial differential equation in (2.9) and $\left(\hat{x}(\cdot),\hat{u}(\cdot)\right)$ be an extremal flow. By (2.4) we see that if $\lambda=0$, then $H(\lambda,x)=0$. If for some $t\in[0,T]$, $B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)=0$, then in (2.19) let $\hat{u}(t)=0$. Thus we have

$$\hat{u}(t)=\begin{cases}-\sqrt{-\beta^{-1}c^{\mathrm{T}}x(t)}\dfrac{B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)}{\left\|B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\right\|}, & B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)\neq 0,\\ 0, & B^{\mathrm{T}}v_x\left(t,\hat{x}(t)\right)=0,\end{cases} \tag{2.21}$$

which is an optimal feedback control to the problem $(\mathcal{P}_1)$.

Example 2.1. Given a function $P(x)\in C^1\left(R^1\right)$, consider the following optimal problem:

$$\begin{cases}\min P\left(x(1)\right)\\ \text{s.t. }\ \dot{x}(t)=u(t),(0)=1,\\ -x(t)+u^2(t)\leq 0,\ x(t)\in R^1,\ u(t)\in R^1,\ t\in[0,1].\end{cases} \tag{2.22}$$

where, for the data corresponding to problem $(\mathcal{P}_1)$, we see that $A=0$, $B=1$, $c=-1$, $\beta=1$.

For $x\in R^1$, $x\geq 0$, define

$$S(x)=\left\{u\in R^1:u^2\leq x\right\}. \tag{2.23}$$

We have

$$\min_{u\in S(x)}\lambda u=-\sqrt{x}\left|\lambda\right|. \tag{2.24}$$

The corresponding partial differential equation in (2.9) is

$$v_t(t,x)-\sqrt{x}\left|v_x(t,x)\right|=0, \tag{2.25}$$

with the boundary condition $v(1,x)=P(x)$. A numerical method can be used to find a viscosity solution to the partial differential equation in (2.25) ([10]).

By Theorem 2.2, we can find an extremal flow by the following differential equation

$$\dot{\hat{x}}(t)=\hat{u}(t),\ \textit{a.e.}\ t\in[0,1], \tag{2.26}$$

where

$$\hat{u}(t)=\begin{cases}-\sqrt{\hat{x}(t)}, & v_x(t,\hat{x}(t))>0,\\ \sqrt{\hat{x}(t)}, & v_x(t,\hat{x}(t))<0,\\ 0, & v_x(t,\hat{x}(t))=0,\end{cases} \tag{2.27}$$

which is an optimal feedback control to the problem in (2.22) and can be obtained numerically ([11]) as follows.

Let $w(t,x)$ be a solution of the partial differential equation in (2.25). We may find a discrete solution of (2.26) with the feedback control presented in (2.27) in the following algorithm. For given positive integer *N*, let $i=1,2,\cdots,N$ and $t_0=0$, $x_0=1$. Solve the following discrete equation:

$$T(x_i-x_{i-1})=Nu_{i-1}, \tag{2.28}$$

$$u_{i-1}=\begin{cases}-\sqrt{x_{i-1}}, & v_x(t_{i-1},x_{i-1})>0,\\ \sqrt{x_{i-1}}, & v_x(t_{i-1},x_{i-1})<0,\\ 0, & v_x(t_{i-1},x_{i-1})=0.\end{cases} \tag{2.29}$$

3. A Convergent Result on the Optimal Value of the Problem $(\mathcal{P}_2)$

In this section for the problems $(\mathcal{P}_2)$ we assume that $m=n$ and the matrix B is invertible. We consider the following constraint optimal control problem:

$$(\mathcal{P}_2):\begin{cases}\text{Minimize } P(x(T)),\\ \text{s.t. } \dot{x}(t)=Ax(t)+Bu(t),\ x(0)=a\in R^n,\\ \quad c^{\mathrm{T}}x(t)\le 0,\\ \quad u(t)\in R^n,\ x(t)\in R^n,\ t\in[0,T]\end{cases} \tag{3.1}$$

where the cost function $P(x)$ is continuously differentiable on R^n. The matrices $A\in R^{n\times n}$, $B\in R^{n\times n}$ and the vector $c\in R^n$ are given in the linear control system in (3.1).

In the following, for a given positive number β, the optimal value of the problem $(\mathcal{P}_1)$ is denoted by V_β and the optimal value of the problem $(\mathcal{P}_2)$ is denoted by $\hat{V}$.

Lemma 3.1. 1) For every given $\beta>0$, $V_\beta\ge\hat{V}$. 2) If $\beta_1\ge\beta_2>0$, then $V_{\beta_1}\ge V_{\beta_2}$.

Proof: Firstly, let $(x(\cdot),u(\cdot))$ be an admissible pair of the problem $(\mathcal{P}_1)$. Note that the matrices A,B and the vector c,a appearing in $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are the same. It follows from the fact $c^{\mathrm{T}}x(t)+\beta u^{\mathrm{T}}(t)u(t)\le 0$, $t\in[0,T]$ that $c^{\mathrm{T}}x(t)\le 0$, $t\in[0,T]$. Thus $(x(\cdot),u(\cdot))$ is also an admissible pair of the problem $(\mathcal{P}_2)$. Consequently, $V_\beta \ge \hat{V}$.

Secondly, let $(x(\cdot),u(\cdot))$ be an admissible pair of the problem $(\mathcal{P}_1)$ with β_1. Note that the matrices A,B and the vector c,a appearing in $(\mathcal{P}_1)$ do not depend on the parameter β. Noting $\beta_1\ge\beta_2>0$, it follows from the fact $c^{\mathrm{T}}x(t)+\beta_1 u^{\mathrm{T}}(t)u(t)\le 0$, $t\in[0,T]$ that $c^{\mathrm{T}}x(t)+\beta_2 u^{\mathrm{T}}(t)u(t)\le 0$, $t\in[0,T]$. Thus, $(x(\cdot),u(\cdot))$ is also an admissible pair of the problem $(\mathcal{P}_1)$ with β_2. Consequently, $V_{\beta_1}\ge V_{\beta_2}$. The lemma is proved.

By Lemma 3.1, if $\beta_k, k=1,2,\cdots$ is a decrease series of positive numbers satisfying $\beta_k\to 0$ when $k\to+\infty$, then there exists a number V^* such that

$$\lim_{k\to+\infty} V_{\beta_k}=V^*\ge\hat{V}. \tag{3.2}$$

In the following we need to show that $V^*=\hat{V}$.

Theorem 3.1. Let $\beta_k, k=1,2,\cdots$ be a decrease series of positive numbers satisfying $\beta_k\to 0$ when $k\to+\infty$. Let $(\hat{x},\hat{u})$ be an optimal pair of the problem $(\mathcal{P}_2)$ satisfying $c^{\mathrm{T}}a<0$. Then

$$\lim_{k\to+\infty} V_{\beta_k}=\hat{V}. \tag{3.3}$$

Proof: Given $\epsilon>0$. Noting $c^{\mathrm{T}}\hat{x}(T)\le 0$, inside a sufficiently small ball which is centered at $\hat{x}(T)$ we can find a point $\overline{x}$ such that $c^{\mathrm{T}}\overline{x}<0$ such that

$$\left|P(\overline{x})-P(\hat{x}(T))\right|<\epsilon, \tag{3.4}$$

also noting that $P(x)$ is continuous.

Since $c^{\mathrm{T}}a<0$, $c^{\mathrm{T}}\overline{x}<0$ and the set $\{x:c^{\mathrm{T}}x\le 0\}$ is convex, we can define a linear function denoted by $\overline{x}(t)$ such that the line $\{\overline{x}(t),\in[0,T]\}$ is contained in $\{x:c^{\mathrm{T}}x<0\}$ satisfying $\overline{x}(0)=a$, $\overline{x}(T)=\overline{x}$, *i.e.*

$$\overline{x}(t)=\left(1-\frac{t}{T}\right)a+\frac{t}{T}\overline{x},\ t\in[0,T].$$

Noting that the matrix B is convertible, now we get a control $\overline{u}(t)=B^{-1}\{\dot{\overline{x}}(t)-A\overline{x}(t)\}$, $t\in[0,T]$, *i.e.*

$$\dot{\overline{x}}(t)=A\overline{x}(t)+B\overline{u}(t),\ t\in[0,T],\ \overline{x}(0)=a. \tag{3.5}$$

We see that the linear function $c^{\mathrm{T}}\overline{x}(t)$ is continuous. It follows $c^{\mathrm{T}}\overline{x}(t)<0$, $t\in[0,T]$ that there is a negative number γ such that

$$\max_{t\in[0,T]} c^{\mathrm{T}}\overline{x}(t)\le\gamma<0. \tag{3.6}$$

On the other hand, since $\overline{u}(t)=B^{-1}\{\dot{\overline{x}}(t)-A\overline{x}(t)\}$, $t\in[0,T]$ is bounded, we have a positive number L such that

$$\max_{t\in[0,T]} \overline{u}^{\mathrm{T}}(t)\overline{u}(t)\le L. \tag{3.7}$$

Then, by (3.6), (3.7), noting $\beta_k\to 0$, for a sufficiently large positive integer K, in the decrease series $\{\beta_k, k=1,2,\cdots\}$, for $k>K$, the positive number $\beta_k>0$ is sufficiently small such that

$$c^{\mathrm{T}}\bar{x}(t)+\beta_k\bar{u}^{\mathrm{T}}(t)\bar{u}(t)<\gamma+\beta_k L<0,\ t\in[0,T]. \tag{3.8}$$

Thus by (3.7), (3.8), $(\bar{x}(t),\bar{u}(t)), t\in[0,T]$ is an admissible pair of the problem $(\mathcal{P}_2)$ with respect to β_k, $k>K$. Noting $P(\hat{x}(T))=\hat{V}$, $P(\bar{x}(T))\geq V_{\beta_k}$, $k>K$ and $\bar{x}(T)=\bar{x}$, by Lemma 3.1 and (3.4), for $k>K$, we have

$$0\leq V_{\beta_k}-\hat{V}=V_{\beta_k}-P(\hat{x}(T))\leq P(\bar{x}(T))-P(\hat{x}(T))=P(\bar{x})-P(\hat{x}(T))<\epsilon. \tag{3.9}$$

Since ϵ is an arbitrary positive number, we have proved

$$\lim_{k\to+\infty}V_{\beta_k}=\hat{V}.$$

The theorem has been proved.

4. Conclusion

It is well-known that in general, it is hard to find a state constrained optimal control problem. In this paper, we study an auxiliary optimal control problem for an approximation approach to the optimal objective value of a state constrained optimal control problem. We also suggest a computational approach to deal with this process by partial differential equations. By more works in future, we may consider more state constrained optimal control problems by this mathematical method.

Conflicts of Interest

The author declares no conflicts of interest.

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